Substituting the value for w given by the second equation into the first gives

$$-z = \lambda^2 z.$$

We know that $z \neq 0$ (if $z = 0$, then the system of equations implies that $w = 0$ as well), so we divide by z to get

$$-1 = \lambda^2.$$

Therefore, $\lambda = i$ or $\lambda = -i$. If $\lambda = i$, the system of equations gives $-z = iw$, which implies that $z = -iw$. Therefore, i is an eigenvalue of R with corresponding eigenvectors

$$\{(w, -iw) : w \in \mathbf{C}\}.$$

If $\lambda = -i$, the system of equations gives $-z = -iw$, which implies that $z = iw$. Therefore, $-i$ is an eigenvalue of R with corresponding eigenvectors

$$\{(w, iw) : w \in \mathbf{C}\}.$$

An operator on $\mathbf{C}^2$ has at most $\dim \mathbf{C}^2 = 2$ distinct eigenvalues by Corollary 5.9. We have already found 2 distinct eigenvalues, so there cannot be more eigenvalues and corresponding eigenvectors of R. The question did not ask about the operator $R \in \mathcal{L}(\mathbf{R}^2)$ because that operator has no real eigenvalues; 90° rotation of a nonzero vector in $\mathbf{R}^2$ never equals a scalar multiple of itself.

4. (a) Let A denote the given matrix of T. We begin by computing some powers of A:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 11 & 4 & -6 \\ 4 & 17 & -12 \\ -2 & -4 & 15 \end{bmatrix}.$$

We want to find a linear dependence between these matrices. By inspection (and some trial-and-error), we see that T satisfies the polynomial $p(x) = x^2 + 2x - 15$. In other words, $p(T) = T^2 + 2T - 15I = 0$.

To show that $p(x)$, of degree 2, has minimal degree among all nonzero polynomials satisfied by T, it suffices to show that T does not satisfy any nonzero polynomial of degree 0 or 1. Suppose that T did. Then there exists some nonzero polynomial $q(x)$ of degree 0 or 1 such that $q(T) = a_0 I + a_1 T = 0$ for some $a_0, a_1 \in \mathbf{R}$. If $a_1 = 0$, we have $a_0 I = 0$, which is possible if and only if $a_0 = 0$. However, this implies that $q(x) = 0$, which contradicts the assumption that $q(x)$ is nonzero. If $a_1 \neq 0$, we have $T = -\frac{a_0}{a_1} I$, which means that T is a scalar multiple of the identity. Any $v \in \mathbf{R}^3$ is an eigenvector for a scalar multiple of the identity, but $(1, 0, 0)$ is not an eigenvector for T, so T cannot be a scalar multiple of the identity.

T does not satisfy any polynomial of degree < 2, so the degree of $p(x) = x^2 + 2x - 15$ is minimal among the degrees of all nonzero polynomials satisfied by T.

(b) We know that $p(T) = T^2 + 2T - 15I = 0$, which can be factored into $(T + 5I)(T - 3I) = 0$. Applying this linear operator to $v \in V$, we get $(T + 5I)(T - 3I)v = 0v = 0$. Therefore, $v \in \operatorname{null}(T + 5I)$, which implies that -5 is an eigenvalue, and/or $v \in \operatorname{null}(T - 3I)$, which implies that 3 is an eigenvalue. -5 and 3 are the only possible eigenvalues of T, which we can verify by finding any corresponding eigenvectors.

[redacted]

(c) If -5 is an eigenvalue of T, then

$$\begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -5 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

gives

$$\begin{aligned} 2x - 2y + 3z &= -5x \\ -2x - y + 6z &= -5y \\ x + 2y &= -5z. \end{aligned}$$

Moving all terms to the left and multiplying the third equation by -1, we get

$$\begin{aligned} 7x - 2y + 3z &= 0 \\ -2x + 4y + 6z &= 0 \\ -x - 2y - 5z &= 0. \end{aligned}$$

Adding these equation together, we get $4x + 4z = 0$, which implies that $z = -x$. Substituting this into the first equation above, we get $7x - 2y + 3(-x) = 4x - 2y = 0$, which implies that $y = 2x$. Therefore, -5 is an eigenvalue of T with corresponding eigenvectors

$$\{(x, 2x, -x) : x \in \mathbf{R}\}.$$

If 3 is an eigenvalue of T, then

$$\begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

gives

$$\begin{aligned} 2x - 2y + 3z &= 3x \\ -2x - y + 6z &= 3y \\ x + 2y &= 3z. \end{aligned}$$

Moving all terms to the left, we get

$$\begin{aligned} -x - 2y + 3z &= 0 \\ -2x - 4y + 6z &= 0 \\ x + 2y - 3z &= 0. \end{aligned}$$

These equations are scalar multiples of each other, and each one implies that $z = \frac{x+2y}{3}$. Therefore, 3 is an eigenvalue of T with corresponding eigenvectors

$$\left\{ \left(x, y, \frac{x+2y}{3} \right) : x, y \in \mathbf{R} \right\}.$$

We see that $v_1 = (1, 2, -1)$ is an eigenvector corresponding to eigenvalue -5, and $v_2 = (3, 0, 1)$ and $v_3 = (2, -1, 0)$ are eigenvectors corresponding to eigenvalue 3. We have found 3 linearly independent eigenvectors of T, so (v_1, v_2, v_3) is a basis of eigenvectors of T.